

EC2020-Fall 2009
Problem Set 1 solutions

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MWG 1.B.1

SUPPOSE \succeq SATISFIES COMPLETENESS AND TRANSITIVITY.

SHOW THAT

$$x \succ y \succeq z \Rightarrow x \succ z$$

IF $x \not\succeq z$ THEN $z \succeq x$.

IF $x \succ y \succeq z$ THEN WE HAVE

$$y \succeq z \succeq x \Rightarrow y \succeq x$$

BUT $x \succ y \Rightarrow y \not\succeq x$ A CONTRADICTION. \square

MWG 1.B.2

SUPPOSE \succeq COMPLETE AND TRANSITIVE

(I) SHOW THAT \succ IS IRREFLEXIVE AND TRANSITIVE.

• BY COMPLETENESS $x \succeq x \forall x$.
 BUT $x \succeq x \Leftrightarrow x \not\succeq x$
 $\Rightarrow \succ$ IRREFLEXIVE $\forall x$

• SAY $x \succ y \succ z$ AND $x \not\succeq z \Rightarrow z \succeq x$ (3)
 FROM THE DEF OF \succ
 THAT WE HAVE $x \succeq y$ (2) $y \succeq z$ (1)
 $x \not\succeq y$ $y \not\succeq z$

FROM (1) + (3) AND TRANSITIVITY OF \succeq $y \succeq x$ (4)

BUT $x \succ y \Rightarrow y \not\succeq x$, A CONTRADICTION \square

(II) SHOW THAT \sim IS REFLEXIVE, TRANSITIVE, SYMMETRIC.

• SINCE \succeq COMPLETE, $x \succeq x$ AND $x \not\succeq x \Rightarrow x \sim x \square$

• IF $x \sim y \sim z$, THEN

$x \succeq y \succeq z \Rightarrow x \succeq z$ (1)

AND $z \succeq y \succeq x \Rightarrow z \succeq x$ (2)

(1) + (2) $\Rightarrow x \sim z \square$

• IF $x \sim y$ THEN $x \succeq y$ AND $y \succeq x \Rightarrow y \sim x. \square$

MWG 1.B.3

LET $f: \mathbb{R} \rightarrow \mathbb{R}$ WITH $f(x) > f(y) \Leftrightarrow x > y \quad \forall x, y \in \mathbb{R}$.

LET $u: X \rightarrow \mathbb{R}$ REPRESENT \succsim .

SHOW THAT $f(u(\cdot))$ ALSO REPRESENTS \succsim .

u REPRESENTS \succsim MEANS

$$u(x) \geq u(y) \Leftrightarrow x \succsim y$$

SUPPOSE $x \succsim y$. THEN $u(x) > u(y)$ OR $u(x) = u(y)$.

IF $u(x) = u(y)$ THEN $f(u(x)) = f(u(y))$.

IF $u(x) > u(y)$ THEN $f(u(x)) > f(u(y))$.

THIS $x \succsim y \Rightarrow f(u(x)) \geq f(u(y))$.

SUPPOSE $f(u(x)) \geq f(u(y))$. THEN $f(u(x)) = f(u(y))$ OR $f(u(x)) > f(u(y))$.

IF $f(u(x)) = f(u(y))$, MONOTONICITY OF $f \Rightarrow u(x) = u(y)$.

SINCE u REPRESENTS $\succsim \Rightarrow x \succsim y$.

IF $f(u(x)) > f(u(y))$, MONOTONICITY $\Rightarrow u(x) > u(y)$.

SINCE u REPRESENTS $\succsim \Rightarrow x \succsim y$. \square

MWG 1.B.4

SUPPOSE ① $u(x) = u(y) \Rightarrow x \sim y$ AND

② $u(x) > u(y) \Rightarrow x \succ y$

SHOW THAT

$$u(x) \geq u(y) \Leftrightarrow x \succeq y.$$

N.B. IF $A \Rightarrow B$ THEN $\neg B \Rightarrow \neg A$ [THIS IS THE "CONTRADICTION"]

APPLYING \Rightarrow ① AND ② WE HAVE

①' $x \succ y$ OR $x \prec y \Rightarrow u(x) \neq u(y)$

②' $x \succeq y \Rightarrow u(x) \leq u(y)$

\therefore ① + ②' \Rightarrow

$$u(x) \geq u(y) \Rightarrow x \succeq y \quad \text{③}$$

REVERSING THE ROLES OF x, y IN ②' GIVES

$$x \succeq y \Rightarrow u(x) \geq u(y) \quad \text{④}$$

③ AND ④ TOGETHER GIVE $x \succeq y \Leftrightarrow u(x) \geq u(y)$
 \square

MWG 1.B.5

- SUPPOSE THAT X IS FINITE AND \succeq IS A RATIONAL PREFERENCE RELATION ON X . SHOW THAT THERE IS A UTILITY FUNCTION $U: X \rightarrow \mathbb{R}$ REPRESENTING \succeq .

• TO START, SUPPOSE THAT NO TWO ELEMENTS OF X ARE INDIFFERENT.

- LET $X = \{x_1, x_2, \dots, x_n\}$. SINCE \succeq IS COMPLETE AND TRANSITIVE WE CAN ORDER ELEMENTS OF X FROM LEAST PREFERRED TO MOST PREFERRED.

- LET $u(x) = \text{RANK OF } x$. THAT IS, THE UTILITY OF THE LEAST PREFERRED ELEMENT IS 1, OF THE NEXT ELEMENT 2, ETC.

THEN $u(x) > u(y) \Leftrightarrow x \succ y$ SO
 u REPRESENTS \succeq ON X .

- NOW SUPPOSE THAT SOME ELEMENTS OF X ARE INDIFFERENT. CONSTRUCT $X^* \subset X$ SUCH THAT ONLY ONE MEMBER OF INDIFFERENT SETS IS RETAINED.

- CONSTRUCT u ON X^* AS ABOVE.

- EXTEND u TO X BY ASSIGNING ALL MEMBERS OF ANY INDIFFERENT SET THE SAME u .

□

MWG 1.C.1

$$(B, C(\cdot)) \quad \beta = \{ \{x, y\}, \{x, y, z\} \} \text{ AND} \\ C(\{x, y\}) = \{x\}$$

SAY $C(\cdot)$ SATISFIES W.A.R.P. THEN

IF $x \in C(\{x, y\})$ THEN FOR ANY B
WITH $\{x, y\} \subset B$ AND $y \in C(B)$,
WE MUST HAVE $x \in C(B)$.

THUS, $C(\{x, y, z\})$ MUST EITHER

(i) NOT CONTAIN y
OR

(ii) CONTAIN BOTH y AND x

IF π SATISFIES W.A.R.P.

$$\Rightarrow C(\{x, y, z\}) \subset \{ \{x\}, \{z\}, \{x, y\}, \{x, z\}, \{x, y, z\} \}$$

BUT IF $C(\{x, y, z\}) \subset \{ \{x, y\}, \{x, y, z\} \}$

THEN W.A.R.P. REQUIRES

$$C(\{x, y\}) \subset \{ \{y\}, \{x, y\} \}.$$

THUS, IF W.A.R.P. IS SATISFIED

$$C(\{x, y, z\}) \subset \{ \{x\}, \{z\}, \{x, z\} \}$$

□

MWG 1.D.1

THIS IS EASY IF YOU KEEP YOUR CHOICE
STRUCTURE VERY SIMPLE.

$$\beta = \{ \{ x, y, z \} \}$$

$$C(\{ x, y, z \}) = \{ x \}$$

THIS IS CONSISTENT WITH $x \succ y \succ z$
AND $x \succ z \succ y$

MWG 1.D.3

$$X = \{x, y, z\}$$

$$\beta = \{ \{x, y\}, \{y, z\}, \{x, z\}, \{x, y, z\} \}$$

$$C(\{x, y\}) = \{x\}, C(\{y, z\}) = \{y\}, C(\{x, z\}) = \{z\}$$

• SUPPOSE $C(\{x, y, z\}) = \{x\}$

THEN W.A.R.P. $\Rightarrow x \in C(\{x, z\}) = \{z\}$

SO W.A.R.P. IS VIOLATED.

SIMILARLY FOR $C(\{x, y, z\}) = \{y\}$ or $\{z\}$.

• SUPPOSE $C(\{x, y, z\}) = \{x, y\}$

THEN W.A.R.P. $\Rightarrow y \in C(\{y, z\})$

SO W.A.R.P. IS VIOLATED.

SIMILARLY FOR $C(\{x, y, z\}) = \{x, z\}$ or $\{y, z\}$.

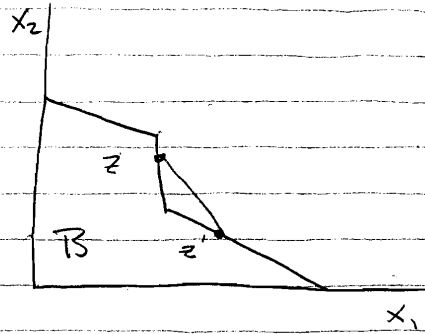
• SUPPOSE $C(\{x, y, z\}) = \{x, y, z\}$

THEN W.A.R.P. $\Rightarrow z \in C(\{y, z\})$

SO W.A.R.P. IS VIOLATED



MWG 2.D.4



$z, z' \in B$, BUT $\frac{1}{2}z + \frac{1}{2}z' \notin B$
 $\Rightarrow B$ NOT CONVEX

MWG 2.E.1

$$x_1(p, w) = \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1}$$

$$x_2(p, w) = \frac{p_3}{p_1 + p_2 + p_3} \frac{w}{p_2}$$

$$x_3(p, w) = \frac{\beta p_1}{p_1 + p_2 + p_3} \frac{w}{p_3}$$

$$x(p, w) \text{ is } H\phi \Rightarrow x(\alpha p, \alpha w) = x(p, w) \\ \forall \alpha, p, w, \alpha > 0$$

$$\text{But } x_1(\alpha p, \alpha w) = \frac{\alpha p_2}{\alpha(p_1 + p_2 + p_3)} \cdot \frac{\alpha w}{\alpha p_1} = x_1(p, w)$$

Similarly x_2 ,

$$\text{For } x_3(\alpha p, \alpha w) = \frac{\beta \alpha p_1}{\alpha(p_1 + p_2 + p_3)} \cdot \frac{\alpha w}{\alpha p_3} = x_3(p, w)$$

So x is $H\phi$ \square

Walras Law requires

$$p \cdot x = w$$

$$\Rightarrow \frac{p_2 + p_3 + \beta p_1}{p_1 + p_2 + p_3} \cdot w = w$$

This is true iff $\beta = 1$. \square

MWG 2.E.4

$$x_2(p, \alpha w) = \alpha x_1(p, w) \quad \alpha > 0, \forall p$$

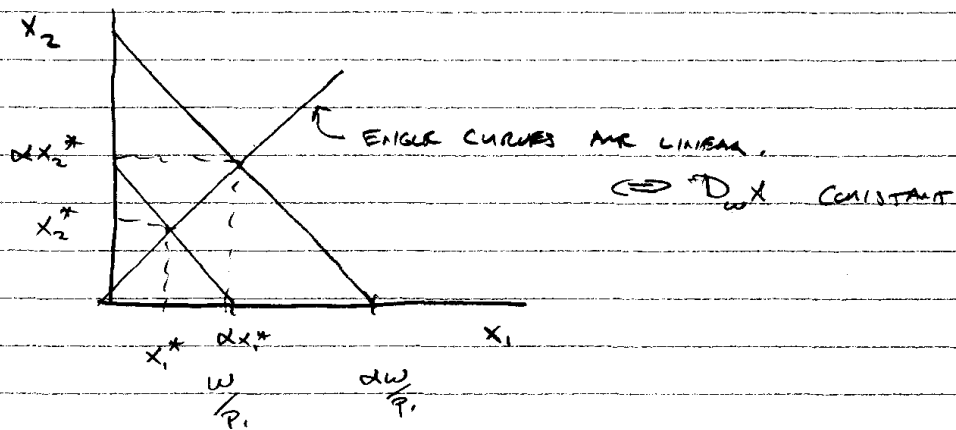
$$\Rightarrow \frac{x_2(p, \alpha w)}{\alpha w} = \frac{x_1(p, w)}{w}$$

$$\Rightarrow \frac{d}{dw} \frac{x_2(p, w)}{w} = 0 \quad \text{i.e. RATIO } \frac{x_2}{w} \text{ CONSTANT}$$

$$\Rightarrow \frac{w \frac{d}{dw} x_2 - x_2}{w^2} = 0$$

$$\Rightarrow w \cdot \frac{dx_2}{dw} = x_2$$

$$\Rightarrow \frac{w}{x_2} \cdot \frac{dx_2}{dw} = 1 \quad (\Leftrightarrow) \quad \epsilon_{x_2, w} = 1 \quad \square$$



MWG 2.F.1

I

DEF 2.F.1: W.A.R.P. IFF FOR ANY
 $(p, w), (p', w')$:

IF $p \cdot x(p', w') \leq w$ (A) THEN
 AND $x \neq x'$ (B)
 THEN $p' \cdot x(p, w) > w'$ (C)

II

DEF 1.C.1: W.A.R.P. \Leftrightarrow IF FOR SOME $B \in \beta$ WITH $x, y \in B$
 WE HAVE $x \in C(B)$, THEN FOR
 ANY B' WITH $x, y \in B'$ WE MUST
 HAVE $x \in C(B')$.

SHOW THAT II \Rightarrow I

LET $B(p, w) = \{x \mid p \cdot x \leq w\}$

SUPPOSE (A) $x(p, w) = C(B(p, w))$ AND

(1) $x(p', w') \in B(p, w)$

(2) $x(p', w') = C(B(p', w'))$

(3) $x \neq x'$

THEN II \Rightarrow (4) $x(p, w) \notin B(p', w')$

(1) AND (2) HOLD BY DEF OF $x(p, w)$.

(1), (3), (4) CORRESPOND EXACTLY TO (A) (B) (C)
 (WITH SOME CHANGE OF NOTATION).

THUS II \Rightarrow I

I $\not\Rightarrow$ II SINCE I CAN IMAGINE SITUATIONS WITH
 CONSTRAINT SETS THAT ARE NOT BUDGET SETS
 WHERE II WOULD APPLY, BUT I WOULD NOT.

MWG 2.F.2

$$w = 8$$

$$\begin{array}{ll} p_1 = (2, 1, 2) & x_1 = (1, 2, 2) \\ p_2 = (2, 2, 1) & x_2 = (2, 1, 2) \\ p_3 = (1, 2, 2) & x_3 = (2, 2, 1) \end{array}$$

Show $x_3 \succsim^* x_2 \succsim^* x_1 \succsim^* x_3$

$$x_3 \succsim^* x_2 \iff p_3 \cdot x_2 \leq 8$$

$$\text{But } p_3 \cdot x_2 = 1 \cdot 2 + 2 \cdot 1 + 2 \cdot 2 = 8 \quad \square$$

$$x_2 \succsim^* x_1 \iff p_2 \cdot x_1 \leq 8$$

$$\text{But } p_2 \cdot x_1 = 2 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 = 8 \quad \square$$

$$x_1 \succsim^* x_3 \iff p_1 \cdot x_3 \leq 8$$

$$\text{But } p_1 \cdot x_3 = 2 \cdot 1 + 1 \cdot 2 + 2 \cdot 2 = 8 \quad \square$$

\square