Caller Number Five:
*Timing Games that Morph from One Form to Another**

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Abstract

There are two varieties of timing games in economics: Having more predecessors helps in a war of attrition and hurts in a pre-emption game. This paper introduces and explores a spanning class with *rank-order payoffs* that subsumes both as special cases. Given unobserved actions and complete information, we identify equilibria with a rich enough structure to capture a wide array of economic and social timing phenomena — shifting between phases of slow and explosive entry.

Inspired by auction theory, we first show how the symmetric Nash equilibria are each equivalent to a different "potential function". This device yields rapid existence and characterization results. The Descartes Rule of Signs, eg., bounds the number phase transitions. We describe how adjacent timing game phases interact: Wars of attrition are truncated, and pre-emptive atoms swelled. We bound the number of equilibria, and compute the payoff and duration of each equilibrium.

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1 Introduction

Suppose that a radio call-in show awards Stones tickets to the seventy-seventh caller. If the number of other potential callers is known, and if waiting to call inflicts opportunity costs on listeners, when should they call? Intuitively, players initially strategically benefit from the delay, but eventually succumb to a fear of missing out. How long will the game last? What economic lessons can be gleaned from players' equilibrium timing behaviour?

Timing models in economics fall into one of two opposing camps. In a *war of attrition*, delay is exogenously costly, and each player prefers that others act before him. The situation is reversed in a *pre-emption game*, where the passage of time is exogenously beneficial, and players wish to pre-empt others. There are, however, many important strategic situations where players prefer to be neither first nor last (fixing the exogenous environment). This class offers many new behavioral phenomena — like multiple periods of slow entry interspersed with sudden rushes.

We observe many situations where rushes seem to arise without forewarning, before and after which life proceeds at a leisurely pace. For instance, why is it that at a party that starts at 8pm, a few guests trickle in early, a large crowd of people arrives exactly at 9pm, and then some stragglers sheepishly show up? Our paper offers a model with rank-dependent payoffs where this can happen, even though nothing exogenous changes about the strategic environment. We characterize the size and timing of such rushes. In particular, we describe how many switches between explosive and slow behavior there can be, how many equilibria are possible, and which payoffs may arise in equilibrium.

We view this as a pure theory paper, towards a foundation for a spanning class of timing games. Still, the motivational radio show example aside, our paper matches some economic applications. For instance, entry into a growing potential new market is often most profitable for early firms after the leader — who struggle with neither market creation nor brand identification. The social phenomenon of fashionable lateness bespeaks a preference for a middling arrival rank. In rush hour one seeks to be early or late.

We develop a comprehensive theory for complete information timing games where the rewards depend on the players' *ordinal stopping ranks.* These rank rewards can be imagined as a reduced form for a richer model, but they afford a sharp focus on the essence of the two strategic forces in timing games. For instance, in a many-player war of attrition, the first stopper earns less than the second, who gets less than the third, etc. The reverse holds in a pre-emption game. In either case, rewards are monotonic in the ordinal stopping ranks. Our formulation extends to non-monotonic rank-rewards.

Returning to the motivational radio show example, one might well imagine that players wait to call, and suddenly enter en masse, jamming the phone lines. What in fact happens is more subtle. Since delaying is explicitly costly, agents are initially locked in a war of attrition. Everyone adopts a mixed strategy, and the chance of winning is ever increasing. Ideally each wants to enter when the *probability* that seventy-six have called is maximal. At that moment, everyone else would do likewise, triggering explosive entry. But the story does not end there. Only one stopper can win, which diminishes the value of the expected prize. The pre-emption moment advances backward in time until everyone is indifferent between pre-empting the entry atom and playing with the mass. Thus, the pre-emption atom 'prematurely' truncates the war of attrition phase. Relative to the direct sum of equilibria from two timing games, agents pre-empt earlier and do so with an excessively large mass. In our paper, both time and size of explosive entry moments are endogenous.

As in the radio call-in show or party arrival examples, we assume unobservable actions. This is a necessary tractability assumption, allowing us to use Nash equilibrium, and adapt potential functions and insights from mechanism design. But our "silent timing games" correspond to economic environments where timing decisions must be made well before the action begins — as with high-tech market entry decisions, or choosing release dates for movies. We also posit discounting and known delay costs. As in most timing game papers, we focus on symmetric equilibria in mixed strategies. This captures an anonymity of play natural in many contexts. We also exclude strategies explicitly depending on focal calendar times or random coordination devices like sunspots.

We show that each such equilibrium tractably admits a unique *potential function* that summarizes play. This concept has many uses in economics (originally, auctions) and the sciences, but in the spirit of all such applications, in our paper, its gradient yields equilibrium expected payoffs (see footnote 5). One example of a potential function is the convex hull of the integrated expected flow payoff. Since it always exists, this yields an immediate proof that our symmetric stationary Nash equilibria exist (Theorem 1).

Given our focus on potential functions, it may come as no surprise that we adapt the "ironing board" of Myerson (1981). While this was not needed for his "regular" auction case with monotonic marginal revenue functions, ironing in our model is used in the "standard" case. Specifically, it is required at each and every explosive entry atom. For another point of comparison, an "ironed" function corresponds to a convexified antiderivative. Myerson seeks the *maximum* revenue auction and thus the convex hull of his integrated marginal revenue function. By contrast, we desire *all* Nash equilibria, and therefore develop a local convexification notion. The convex hull in particular signifies the equilibrium with the highest payoff loss due to delay. Our equilibrium characterization reduces to analyzing all possible potential functions.

In our model, a war of attrition phase obtains only for rising expected payoffs — when strategic and exogenous delay costs conflict. Pre-emptive behaviour is likewise mandated when expected flow payoffs fall, since no conflict is possible. So the slope sign changes of expected payoffs are key. Theorem 2 bounds the number of phases transitions by the underlying deterministic rank reward using the Descartes Rule of Signs; this provides a simple upper bound on the number of phase transitions that binds for some equilibria.

A transition from a war of attrition phase to a pre-emptive explosion (or back) can only occur if expected flow payoffs before (or after) atomic entry coincide with the atomic payoff. As seen in our radio call-in show example, the slope of expected flow payoffs alone does not determine equilibrium play. Rather, the relation between expected flow and atomic rewards matters — and these relate exactly as do marginals and averages (Lemma 2). We build on this insight to deduce that war of attrition phases are "prematurely" truncated in equilibrium and pre-emptive atoms are inflated: In other words, the war ends before expected flow payoffs peak and starts after they trough (Theorem 3).

With multiple phases possible, there are potentially multiple equilibria. Since wars of attrition and pre-emption games must alternate, the question is whether any consecutive pair of them is played. With J such matched pairs, this choice must be made for all of them. Theorem 4 therefore shows that the number of potential Nash equilibria equals 2^J.

In the war of attrition, all rents — namely, the greatest minus the least expected flow rank payoff — are dissipated. This is not true here because rank order payoffs are non-monotonic. As a result, the pre-emption games start before the peak flow expected payoff, when expected flow and average atomic payoffs coincide; thus, the maximal flow rank payoff is not attained in equilibrium. Theorem 5 instead shows that the maximal payoff burn in the game is captured not by a difference of expected flow payoffs, but by a difference of the greatest backward average payoff and the least forward average payoff. Also, the game's expected payoff is at least the minimum of the forward average payoffs. This contrasts with the war of attrition, where the value is the least expected flow payoff.

Our analysis is so tractable that there is a separation of rank payoffs and time costs. Since costs play an important role in determining equilibrium strategies, one might think that not much can be said about the equilibrium without specifying them. But the potential function is derived from the rank payoffs alone, which determines an equilibrium for *any* time costs. Further, the patterns of wars of attrition and explosive phases are identical functions of the stopping probability. For example, if the game starts with an explosive phase, the size of the first atom is independent of costs.

We conclude by briefly considering observable actions. This produces multiple information sets and vastly enriches the set of supportable equilibria (now subgame perfect).

Still, we briefly argue that our main qualitative insight about atom inflation and war of attrition truncation from Theorem 3 remains applicable with a simple refinement.

Maynard Smith (1974) first formalized the war of attrition for theoretical biology: Two animals fight over a fallen prey, the first to give up loses, and fighting is costly for both. With multiple players, payoffs are increasing in the stopping rank. Hendricks, Weiss, and Wilson (1988) have characterized continuous time complete information war of attrition-equilibria, while Bulow and Klemperer (1999) analyzed a generalized N-player war of attrition. Abreu and Pearce (2006) have applied wars of attrition to bargaining.

The pre-emption game has also been studied widely. Very early work focused on tactical duels[1] — two player zero-sum timing games played on a compact time-interval. Two duelists shoot at each other with accuracy increasing in proximity, and they may or may not observe the other's shot. Modern economic examples are aptly captured by the 'Grab-the-Dollar': A player can either grab the money on the table or wait for one more period; meanwhile, the pot increases by one unit. Players want to be the first to take the money, but would rather grab a larger pot. Recent examples are Abreu and Brunnermeier (2003), who model financial bubbles (also with unobserved actions), Levin and Peck (2003, 2005) who look at market entry, and Bouis, Huisman, and Kort (2006) and Argenziano and Schmidt-Dengler (2006) who study N-player investment dynamics.

In independent work, Sahuguet (2006) explores the equilibria of a three player timing game with both pre-emption and attrition features. His payoffs are not rank-dependent. In a recent paper, Laraki, Solan, and Vieille (2005) (LSV) study the existence of equilibria in general timing games; they provide a very compelling argument for the existence of an ϵ-equilibrium in two-person timing game, and an existence argument for two other classes (cumulative and symmetric, as defined in their paper); these existence results, however, do not overlap with our general existence and characterization theorems.[2] Amidst this large literature on timing games, we believe that, our respective works are the first that are neither just a pre-emption game nor just a war of attrition. We hope that it suggests a wider and richer application of timing games in economics.[3] It offers insight into periodic unexpected rushes of uncertain size, followed by relative quiet.

[1]In 1949, the RAND Corporation kick-started the study of duels (*silent timing games*) with a conference with leading economists, statisticians, and economists — for an extensive survey see Karlin (1959).

[2]Our existence results are not subsumed by LSV. Their Theorem 1.2 assumes two players. LSV have other existence results for more than two players, but none applies for us: Our payoffs are not cumulative (Theorem 1.3), nor symmetric (as defined by LSV, Theorem 1.4). Their Theorem 1.5, which may admit ordinal rank-payoffs, requires no time costs or discounting; Also, it secures existence of a Nash equilibrium only if an ϵ-equilibrium exists (which would thus need to be proven separately) for every ϵ.

[3]Shinkai (2000) developed a three-player Stackelberg-type game that fits our rank-payoff formulation: In his framework, quantity pre-emption and learning from predecessors' choices interact to effectively form U-shaped rank rewards. Shinkai, however, does not model the timing decision explicitly.

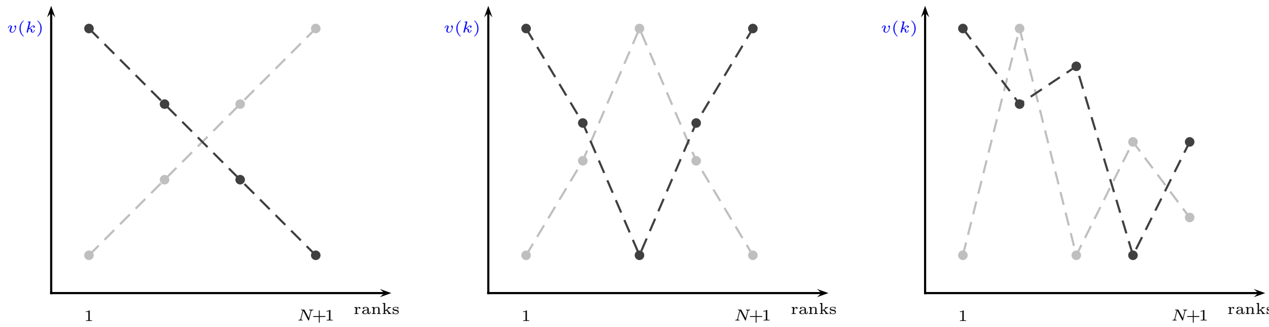

Figure 1: **Plots of rewards structures.** *Left Panel:* A stylized War of Attrition reward structure (gray, higher ranks yield higher rewards), and a stylized pre-emption game reward structure (black, low ranks are better). *Middle Panel:* Hill-shaped reward structure (gray, the some middle rank is best), and an 'avoid-the-crowd' U-shaped reward structure (black, either a very low or very high rank is best). *Right Panel:* Two general reward structures with multiple hills: there are multiple 'locally' optimal ranks.

Overview. In Sections 2 and 3, we outline the model, and derive the potential function notion for the equilibrium analysis. In Section 4, we bound the numbers of equilibria and phase transitions, and show how wars of attrition are truncated and pre-emptive atoms inflated in equilibrium. Section 5 bounds the payoffs and game durations of our equilibria. Section 6 lays out the equilibrium analysis for observable actions.

2 A Model of Timing Games with Rank-Dependent Payoffs

Actions and Strategies. Since we analyze settings where players desire to be neither first nor last, we assume $N+1 \geq 3$ players. Play transpires in continuous time, starting at time $t=0$. Players are identical and have only two decisions: 'to stop' or 'not to stop'; they may stop only once; a stopping decision is irrevocable. Actions are unobservable.

With unobservable actions, there is only one information set. A player's strategy specifies the point in time when he will stop. A mixed strategy is a non-decreasing and right-continuous cdf $G : [0, \infty) \to [0, 1]$, where a player stops with chance $G(t)$ by time t. This strategy space formulation rules out, for instance, "consecutive" atoms.

Payoffs. Upon stopping, a player receives a lump-sum reward that depends on his ordinal stopping rank. This payment is captured in the *reward-function* $v : \{1, \ldots, N{+}1\} \to \mathbb{R}_+$. For instance, in a two-player war of attrition, $v(1) = 0$ and the prize is $v(2) > 0$. In the Caller Number Five game, $v(k) = 0$ for all $k \neq 5$, and the prize is $v(5) > 0$. In general, more predecessors helps in a war of attrition— or $v(k) < v(k+1)$ for all k. In a pre-emption game, the situation is reversed, as more predecessors hurts, or $v(k) \geq v(k{+}1)$ for all k. See Figure 1 for various rank-reward structures.

Agents who stop at the same time equally share the available rank rewards. This is tractable and retains a single information set. It also realistically reflects the anonymity of random stopping. Players don't control their rank order among simultaneous stoppers, and therefore all rank order are equally likely.

Assume then that $k \in \{0, \ldots, N\}$ players have stopped, and $j+1 \in \{1, \ldots, N-k+1\}$ players stop together. Then the *atomic rewards* are the average rank reward $\mathsf{A}(k,j) := (v(k+1)+\cdots+v(k+j+1))/(j+1)$. For instance, in a war of attrition, if both agents stop immediately, then their order is randomly determined, and they share the prize equally.

There are two types of explicit costs: Discounting at the interest rate $r \geq 0$, and exogenous participation costs $c(t)$, with $c(0) = 0$, $\dot{c} > 0$, and $\lim_{t\to\infty} c(t) = \infty$.[4]

Equilibrium. Players are ex ante identical and anonymous. It is then intuitive to explore symmetric strategy Nash equilibria. To avoid a continuum of arbitrary outcomes, we confine attention to equilibria whose cdf G has convex *support* starting at 0. (The support of G is the set of all t with $G(t+\varepsilon) - G(t-\varepsilon) > 0$ for all $\varepsilon > 0$.) To summarize:

E1: *The support of G is a connected interval $[0, T]$ or $[0, \infty)$.*

This restriction is designed to preclude equilibria with explicit periods of silence due to unspecified reasons — calendar time or random holidays (sunspots). But we argue in the paper that it actually embodies a much stronger *stationarity* assumption. We argue in Section 4 that when we relax this assumption a continuum of equilibria arises — each of which is qualitatively identical to the equilibrium that we study.

3 Equilibrium Analysis

In this section, we outline several tools used in equilibrium analysis: necessary conditions for mixed strategies, atomic entry, potential functions, and general existence.

3.1 First Order Conditions for Smooth Entry

Consider a symmetric strategy $G(t)$. If $G(t) = g$, then k of N players independently have chosen to stop with probability $\binom{N}{k} g^k (1-g)^{N-k}$. Hence, if exactly one player enters at time t, then he secures the rank payoff $v(k+1)$. Altogether, expected "flow" or "solo" rewards are

$$\phi(g) := \sum_{k=0}^{N} \binom{N}{k} g^k (1-g)^{N-k} v(k+1). \tag{1}$$

The function ϕ does not depend on the equilibrium and is a primitive of the game.

In any mixed strategy equilibrium, an agent must be indifferent about stopping at any point in time, so that expected flow payoffs are constant on the support. Payoffs are

[4] In a related work, we also explore time benefits, but since the analysis for general rank rewards gets unattractively complicated and we only perform the analysis there for hill- and U-shaped rank rewards.

discounted rewards less discounted costs,

$$e^{-rt}[\phi(G(t)) - c(t)]. \tag{2}$$

Assume $\dot{G}(t)$ exists. Then in equilibrium, payoffs are constant, and equating the *marginal exogenous costs* and *marginal strategic gains* from delay, we get:

$$\dot{c} + r\ (\phi(G) - c) = \dot{G}\phi'(G) \tag{3}$$

We can clearly only solve for $\dot{G} \geq 0$ in any equilibrium if $\dot{c} + r(\phi(G) - c)$ and $\phi'(G)$ share the same sign — namely, if $\phi'(G) > 0$. For there must be a strategic incentive to delay, since one advances in the ranks to greater payoffs. This is true in a war of attrition.

Lemma 1 (The Structure of Equilibria) *Any stationary Nash equilibrium is described by a cdf G consisting solely of atomic jumps and continuously differentiable intervals.*

While a cdf is monotone, and thus almost everywhere differentiable, the jumps may be dense in $(0, 1)$ (maybe the rationals), and there may be non-jump points where G is not differentiable. The proof in the appendix rules out both possibilities.

3.2 Analogy for Atomic Rewards: Average vs. Marginal Revenue

Suppose that the N other players, acting independently, have stopped with chance $G(t) = g$ by time t, at which time each plays with chance $h - g$, where $h > g$. We often refer to $h - g$ as the *atom* or *mass*. Then the chance that players of ranks $k + 1, \ldots, k + j$ stop at time t equals a trinomial coefficient $N!/k!j!(N - k - j)!$ times $g^k(h - g)^j(1 - h)^{N-k-j}$. The expected payoff in this atom, *should one also join,* is then

$$\Lambda(g, h) := \sum_{k=0}^{N} \sum_{j=0}^{N-k} \frac{N!}{k!j!(N - k - j)!} g^k (h - g)^j (1 - h)^{N-k-j} \mathsf{A}(k, j) \tag{4}$$

Thus, $\Lambda(0, h)$ is the payoff of an *initial atom* of size h, and $\Lambda(g, 1)$ the payoff of a *terminal atom* of size $1-g$. When $0 < g < h < 1$, $\Lambda(g, h)$ is the average payoff in the *on-path atom* from g to h, and $\Phi(g) = \int_0^g \phi(s)ds$ the anti-derivative of $\phi(g)$. This motivates:

Lemma 2 *$\Phi(h) - \Phi(g) = (h - g)\Lambda(g, h)$.*

While it is possible to prove this algebraically, it is messy (and omitted). Yet it is intuitive: Independently place each of the N other players into the stopped, atom, and remaining groups, with respective weights $(g, h - g, 1 - h)$. The expected average rank payoff in the 'atom' group is then $(\Phi(h) - \Phi(g))/(h - g)$, by definition of a conditional expectation. But this is how we have defined $\Lambda(g, h)$, and so these measures coincide.

This has a nice illustrative analogue in standard producer theory: When AR and MR denote average and marginal revenue, and q is quantity, then $MR - AR = qAR'(q)$. Differentiating Lemma 2 w.r.t. h directly yields $\phi(h) - \Lambda(g, h) = (h - g)\frac{\partial}{\partial h}\Lambda(g, h)$. This admits an analogous interpretation: $h - g$ is the mass of the atom, and corresponds to the quantity. The expectation $\Lambda(g, h)$ aggregates and averages rewards, and $\phi(h)$ is the derivative of aggregated (non-averaged) rewards. Lemma 2 thus implies that $\phi(\cdot)$ crosses $\Lambda(g, \cdot)$ from above at the local interior maxima of Λ, and from below at the minima.

3.3 Equilibrium, Potential Functions, and Existence

We have already specified that we only consider right-continuous cdfs $G : [0, \infty) \to [0, 1]$ for symmetric stationary Nash equilibria that have convex support, including 0 (labeled as (E1) in Section 2). In any equilibrium, net payoffs are constant along the support of play, and there is no strict incentive to out-wait all other players. Conversely, these are sufficient conditions for a Nash equilibrium. An equilibrium is formally a cdf obeying:

E2: $e^{-rt}[\phi(G(t)) - c(t)]$ *is the same constant for all times in the support of* G *with* $G(t) < 1$*;*

E3: *If* $G(t^*) > G(t^*-)$*, then* $\phi(G(t^*-)) = \Lambda(G(t^*-), G(t^*)) \geq \phi(G(t^*))$ *(equal if* $G(t^*) < 1$*).*

In a symmetric mixed strategy equilibrium, the range $[0, 1]$ of G partitions into subintervals having endpoints $0 = \xi_0 < \xi_1 < \cdots < \xi_k = 1$. Atomic entry (where G jumps) occurs at the points $\{\xi_i\}$, possibly excluding $i = 0$ or $i = k$, and smooth entry (where G' exists) on the intervals $[\xi_i, \xi_{i+1}]$. To find an equilibrium cdf $G(t)$, we thus solve (3) subject to the right boundary conditions, determine atomic jumps so that $(E3)$ holds, and then ensure that the boundary conditions reflect the atomic jumps.

We now turn to an alternative formulation of equilibrium that captures equilibrium conditions in a function. A C^2 function $\Gamma : [0, 1] \to \mathbb{R}_+$ *induces* a strategy G for Φ if

- $\dot{G} = (\dot{c} + r[\Gamma'(G) - c])/\Gamma''(G)$ whenever $\Gamma(G(t)) = \Phi(G(t))$,

- if $\Gamma \neq \Phi$ on a maximal interval (g, h), then $G(\cdot)$ jumps from g to h.

The function $\Gamma : [0, 1] \to \mathbb{R}_+$ is a *potential function*[5] w.r.t. Φ if

[5] Our phrase "potential function" is in the spirit of a harmonic function whose derivatives describe the gradient on a conservative vector field. Closest to our work, in Myerson (1981), the convex hull of integrated "virtual valuations" for the auction is a potential function; its derivatives fix the priority level for allocating the good. Hart and Mas-Colell (1989) may be the first to use the phrase "potential function" in game theory; *differences* of their potential function yielded marginal payoff contributions in a transferable utility game. Our concept bears no relation to the "potential games" literature — eg.,

P1: $\Gamma(0) = 0$, $\Gamma(1) = \Phi(1)$, *and* $\Gamma'(1) \geq \Phi'(1)$*;*

P2: Γ *is monotonically increasing, convex, and continuously differentiable;*

P3: *At each* $\xi \in (0,1)$*, either* $\Gamma(\xi) = \Phi(\xi)$*, or* Γ *is linear in an open interval around* ξ*.*

By properties of the Beta distribution, we see that

$$\Phi(1) = \int_0^1 \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} v(k+1) dx = \frac{1}{N+1} \sum_{k=0}^{N} v(k+1)$$

In other words, $\Gamma(1)$ is the average rank payoff by $(P1)$, while $\Gamma'(1) \geq \phi(1) = v(N+1)$.

Lemma 3 (Equivalence Result) *Fix* Φ*. Any potential function* Γ *induces a unique equilibrium cdf* G*, and any equilibrium cdf* G *is induced by a unique potential function* Γ*.*

The proof is in the appendix. In brief, we argue: If a potential function exists, it yields equilibrium rank payoffs by differentiation. Convexity ensures that the rank payoffs increase (as mandated by increasing time costs). Linear segments in the potential function correspond to atomic entry, whose payoffs are given by the slope of the corresponding linear segment. Since Φ, the anti-derivative of ϕ, is a polynomial, it is arbitrarily smooth; since Γ is continuously differentiable and smooth and either coincides with Φ or is linear, at the join between a smooth and a linear segment the slopes of the smooth and the linear part coincide. Thus the payoffs from the corresponding atom and the payoffs from slow play before and after the atom coincide.

Conversely, a potential function is found by setting $\Phi(G(t)) = \Gamma(G(t))$ whenever $G(t)$ is left-continuous; increasing rank payoffs ensures the convexity of Γ. When G jumps from g to h, there is a linear segment in Γ with endpoints $(g, \Phi(g))$ and $(h, \Phi(h))$; the slope of this segment coincides with the atomic payoff, by Lemma 2. Since atomic and flow payoffs coincide in equilibrium, the slopes at the end points coincide, and Γ is differentiable.

The equivalence lemma is important because it identifies which game fundamentals matter for equilibrium analysis. For instance, costs can only speed up or slow down play. It also allows us to prove theorems more graphically, using potential functions.

Example 1: Caller Number Two of Three. Assume $N+1 = 3$ and $v = (0,1,0)$. Then $\phi(g) = 2g(1-g)$ and $\Phi(g) = g^2(1-2g/3)$. There are exactly two potential functions: First, Γ may initially equal Φ, so that $\Gamma_1(g) = \Phi(g)$ for $g \leq 1/4$ and $\Gamma_1(g) = 3g/8 - 1/24$ for $g > 1/4$. Second, Γ may be initially linear, whereupon it remains linear on $[0,1]$, by convexity, differentiability and $(P3)$: $\Gamma_2(g) = g/3$. These obey the key properties of smoothness, convexity and boundary values: e.g. $\Gamma_2'(1) = 1/3 > \Phi'(1) = 0$.

the potential function in Monderer and Shapley (1996) is a function of the vector of quantities in an IO game. Our potential function maps from a scalar probability.

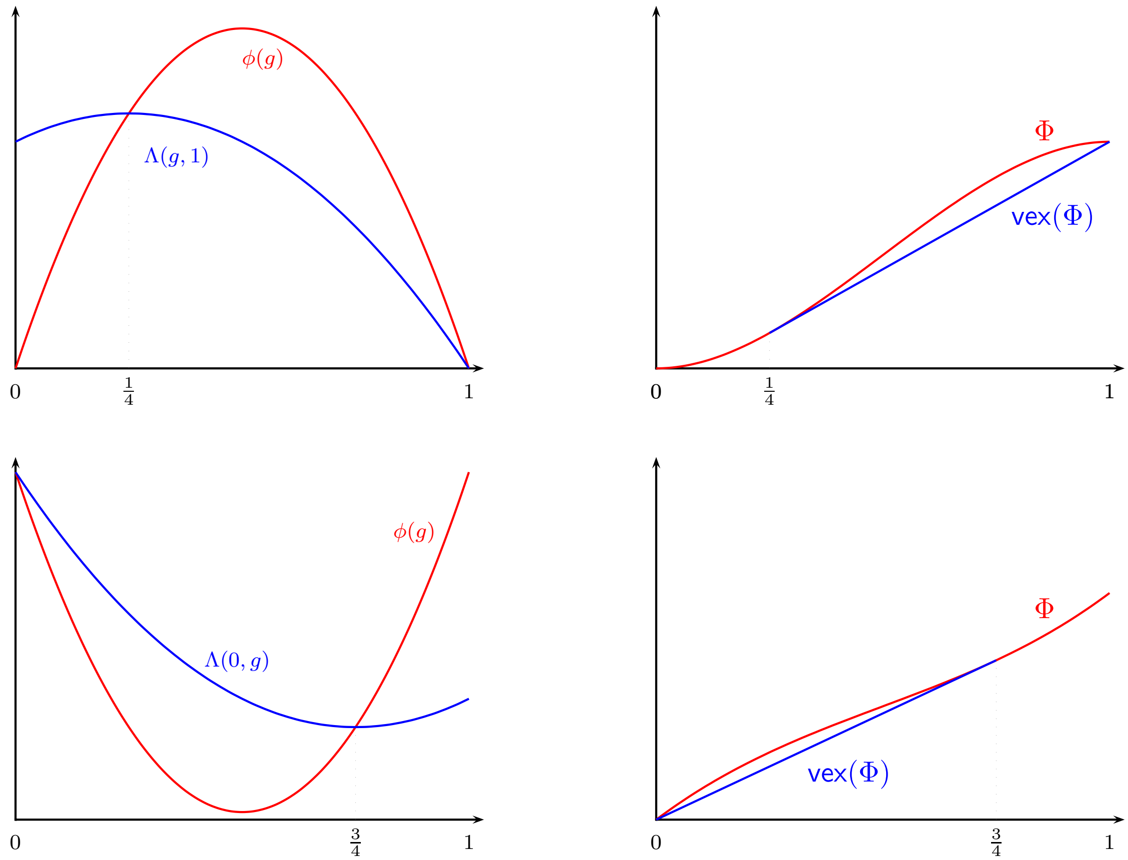

Figure 2: **Examples 1 and 2 from Section 3: Caller 2 of 3 and U-Shaped Rank Payoffs.** The top left panel depicts the flow and average payoffs $\phi(g)$ and $\Lambda(g,1)$ for the Caller 2 of 3 game. The top right panel plots the running integral of payoffs Φ and the potential function $\mathsf{vex}(\Phi)$ identified in the existence Theorem 1. For the U-shaped example, the bottom left figure plots ϕ and $\Lambda(0,g)$ and the bottom right figure plots Φ and the unique potential function $\mathsf{vex}(\Phi)$. The plots also illustrate the theorems later on: Both examples attain the upper bound number two of phases (Theorem 2). Consistent with Theorem 3, the war of attrition is truncated in each case. Just as in Theorem 4, there are two equilibria in the top game (the potential function for the unit jump is not drawn), and one in the bottom game.

Assume delay costs $c(t) = t$ and no discounting. This determines the speed: The first equilibrium involves smooth play described by the ODE $0 = -1 + \dot{G}(t)(1 - 2G(t))$ from (3), with solution $G(t) = 1/2 - 1/2\sqrt{1-2t}$ until $G(t) = 1/4$. At that point $t = 1/4$, a jump to $G = 1$ occurs. The second equilibrium entails simply a time-0 jump to $G = 1$.

Example 2: U-Shaped Rank Payoffs. Assume $N+1 = 3$ and $v = (1, 0, 1)$. Then $\phi(g) = (1-g)^2 + g^2$ and $\Phi(g) = 1/3g^3 - 1/3(1-g)^3$. Here there is a unique potential function $\Gamma_3(g) = 5g/8$ for $g \leq 3/4$ and $\Gamma_3(g) = \Phi(g)$ for $g > 3/4$. Next, solving (3) yields $0 = -1 + 2\dot{G}(t)(2G(t) - 1)$, with solution $2G(t) = 1 + \sqrt{1/4 + 2t}$. Continuous play begins at $t = 0$, with $G(0) = 3/4$. Figure 2 illustrates both examples.

In mechanism design problems, non-monotonic payoff functions are often "ironed" to produce a monotonic function (e.g. Baron and Myerson (1982)). Namely, let $\mathsf{vex}(\Phi)$ be the

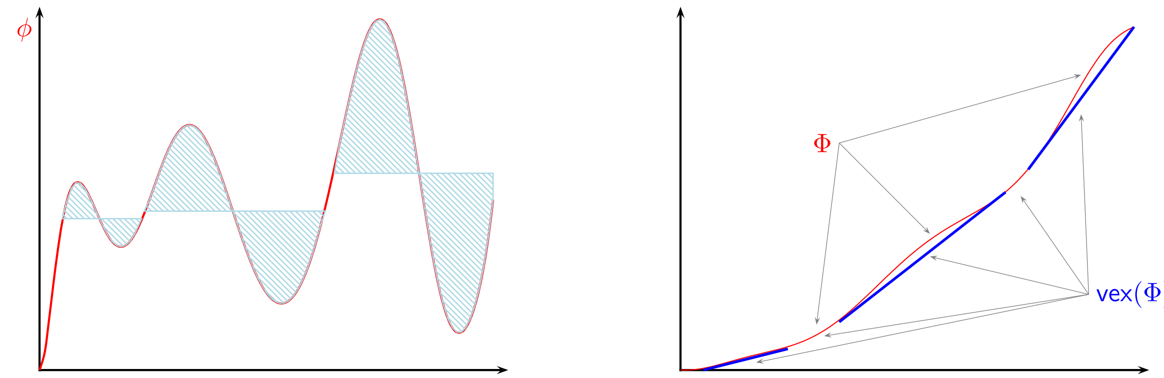

Figure 3: **Ironing ϕ.** The left panel illustrates the ironing procedure on ϕ, the right panel depicts both Φ and the convex hull of Φ, called $\mathsf{vex}(\Phi)$.

convex hull of Φ, i.e. the largest convex function with $\mathsf{vex}(\Phi)(g) \leq \Phi(g)$ for every g. The "ironed" function then is the derivative $\mathsf{vex}(\Phi)'(g)$ (see Figure 3). Our potential functions follow a similar idea. Since exogenous costs are ever-increasing, so must be the expected rank-payoffs. Rank-payoffs however, may decline, and these non-monotonicities must be ironed away. Our potential function describes exactly how this works: its derivative is the rank payoff, its convexity ensures that equilibrium payoffs increase. If the potential function contains a linear segment, then rank payoffs are constant, and since delay is costly, atomic entry must occur.

Theorem 1 *A symmetric mixed strategy equilibrium exists and ends in finite time.*

Proof: First, $\mathsf{vex}(\Phi)$ exists, is a potential function, and thus induces an equilibrium.

In any equilibrium, payoffs are constant on the support at $\phi(0)$. So there exists $\tilde{t} < \infty$ with $\max_g e^{-rt}[\phi(g) - c(t)] < \phi(0)$ after $\tilde{t}$. Delaying beyond $\tilde{t}$ is a dominated strategy, as rewards are discounted or eaten by exogenous delay costs, given $\lim_{t\to\infty} c(t) = \infty$. □

In the Caller Number Two of Three example, $\mathsf{vex}(\Phi)(g) = \Gamma_1(g)$. In the U-shape example, $\Gamma_3(g)$ is the unique potential function, and therefore coincides with $\mathsf{vex}(\Phi)(g)$.

4 Behavioral Properties of Equilibria

4.1 Phases and Phases Transitions

We first bound the number of slope-sign changes of the expected flow rewards.

Lemma 4 (Variation Diminishing Property of Expected Rank Rewards) *Let the slope of rank rewards $v(k)$ change sign m times. Then the slope of expected rewards $\phi(g)$ changes sign at most m times, the number of sign-variations in ϕ is smaller by a multiple of two (including 0), and the signs of the first and last slopes of v and ϕ coincide.*

Proof: The derivative of $\phi(g)$ in g can be rearranged as follows:

$$\phi'(g) = \sum_{k=1}^{N} \binom{N}{k} k g^{k-1} (1-g)^{N-k} (v(k+1) - v(k)).$$

The first differences $v(k+1) - v(k)$ swap their sign m times. Scale ϕ' by $g/(1-g)^N$, and let $a_k := k\binom{N}{k}(v(k+1) - v(k))$ and $z := g/(1-g)$. Then

$$\frac{g}{(1-g)^N}\ \phi'(g) = \sum_{k=1}^{N} k \binom{N}{k} (v(k+1) - v(k)) \left(\frac{g}{1-g}\right)^k = \sum_{k=1}^{N} a_k z^k =: P(z).$$

Obviously, $P(z(g))$ and $\phi'(g)$ enjoy the same number of sign variations, i.e. positive real roots of P. By *Descartes' Rule of Sign*, this number is at most the number of sign changes of its coefficients $a_0, a_1, \ldots, a_N$. Also, if smaller, it is smaller by a multiple of 2.

Finally, $\phi'(0) = v(2) - v(1)$ and $\phi'(1) = v(N+1) - v(N)$, proving the last clause. □

As noted earlier, this paper subsumes and extends two classes of standard timing games: In a *war of attrition*, an exogenous delay cost opposes a strategic incentive to outwait others. The reverse holds in a *pre-emption game*, where delay is exogenously beneficial, and players wish to pre-empt others. We now categorize game phases by their strategic incentives. There is a *war of attrition phase* if $\dot{G}(t+) > 0$ exists and $\phi'(G(t+)) > 0$ on $(\underline{t}, \bar{t})$. A *pre-emptive explosion* obtains if G jumps at t, as $G(t) > G(t-)$.

A *phase transition* occurs at some time t if two distinct timing games obtain in every neighborhood of t. If three game phases obtain, then there are two phase transitions at t. In what follows, we shall drop the term 'phase' from the game descriptions.

Theorem 2 (Phase Transitions)
(a) *Equilibrium play consists solely of an alternating sequence of wars of attrition and pre-emptive explosions. There are no slow pre-emption games, and pre-emptive atoms subsume entire intervals when ϕ is decreasing.*
(b) *There are at most as many phase transitions as sign changes of $v(k) - v(k-1)$.*
(c) *If ϕ has m alternating slope signs, then the maximal number of phase transitions is $m-1$. This bound is attained in equilibrium iff $\mathsf{vex}(\Phi)$ touches every convex portion of Φ.*

Theorem 2 implies that there are no slow pre-emption game phases.[6] Intuitively, we only assume exogenous flow costs of delay, and no benefits, and thus there can be no opposition of strategic costs of delay and exogenous benefits.

[6]Formally, a *slow pre-emption game phase* obtains if $\dot{G}(t+) > 0$ exists and $\phi'(G(t+)) < 0$ on $(\underline{t}, \bar{t})$.

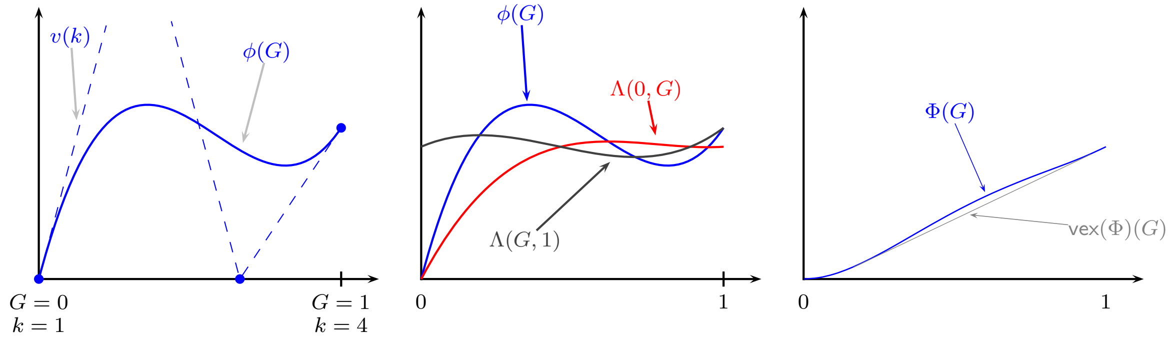

Figure 4: **The Zick-Zack Game**: In this merger of examples 2 and 3, rank payoffs now twice change slope, as $v = (0, \psi, 0, 1)$. If $v(2) = \psi$ (off the graph) is large enough, then the expected flow reward ϕ likewise has both a hill and a valley. Otherwise, ϕ is monotonically increasing. The middle panel plots the expected reward function ϕ and reward functions for initial atoms, $\Lambda(0, g)$ and terminal atoms $\Lambda(g, 1)$. The right panel plots $\Phi(g)$ and $\mathsf{vex}(\Phi)(g)$.

Proof of (a): Expected payoffs are constant along the support of play. Delay is exogenously costly, and so a player's expected rank reward payoff rises over time in equilibrium. If ever $\phi' < 0$, players must stop since delay is both strategically and exogenously costly. So play involves slow war of attrition phases and pre-emptive explosions.

Proof of (b): This follows because ϕ cannot have more interior extrema than v, by Lemma 4 — which also showed that the first and last slope signs of v and ϕ match.

Proof of (c): A phase transition occurs iff Γ switches between locally linear and strictly convex ($\Gamma'' > 0$). The smooth Γ only changes slope when $\Gamma = \Phi$. As a non-linear polynomial, Φ has at most as many strictly convex portions as Γ, with equality iff ($\star$): Γ touches each convex portion of Φ. As $\mathsf{vex}(\Phi)$ is a potential function, this proves sufficiency. Next, assume ($\star$). The smooth Γ includes the unique supporting tangent line between all consecutive convex portions. The unique such potential function is $\mathsf{vex}(\Phi)$. □

One can show that the maximum number of phase transitions is attained only if both the sequence of minima of $\Lambda(0, g)$ and the sequence of maxima of $\Lambda(g, 1)$ are increasing.

Example 3: Zick-Zack. The left panel of Figure 4 depicts the four player game *Zick-Zack*, with rank rewards $v = (0, \psi, 0, 1)$, with $\psi > 0$. Then

$$\begin{aligned}\phi(g) &= 3g(1-g)^2 \cdot \psi + g^3 \cdot 1 \;\text{ and }\; \phi'(g) = 3\psi(2g-1)^2 + 3(1-\psi)g^2, \\ \Phi(g) &= \left(1/4\,(1-(1-g)^4) - g(1-g)^3\right) \cdot \psi + g^4/4.\end{aligned}$$

Analyzing $\phi(g)$, one can see that $\phi(g)$ is monotonic for $\psi \leq 1 =: \underline{\psi}$, even though the underlying rank reward structure v has two slope-sign changes. This illustrates the strict inequality in Lemma 4, by a multiple of two. Then Φ is convex with the unique potential function $\Phi = \mathsf{vex}(\Phi)$; thus there are no phase transitions (Theorem 2 (b)).

If $\psi > \underline{\psi}$, then ϕ has two slope-sign-changes, like v. The quartic polynomial Φ thus has two points of inflection, and $\mathsf{vex}(\Phi)$ must contain at least one linear portion. Hence, there can be at most two phase transitions (Theorem 2 (c)).

Next, $\mathsf{vex}(\Phi)$ touches both the first and second convex portions of Φ for $\underline{\psi} \leq \psi \leq \bar{\psi} := (5+\sqrt{33})/4$. By Theorem 2 ($c$), the associated equilibrium has the maximum number of phase transitions (two): war of attrition, pre-emptive atom, and then war of attrition.

We have shown that ϕ smoothes out rank payoffs relative to v, reducing the number of possible phase transitions below that suggested by a simple examination of the rank payoffs v. Next, one could naïvely imagine that each slope sign change of the smooth function ϕ initiates a phase transition. The naïve equilibrium would be one where a war of attrition obtains iff $\phi' > 0$ and a pre-emption game obtains iff $\phi' < 0$. This is not what happens in equilibrium. First of all, while ϕ may be non-monotonic, the only equilibrium may well be a unique pre-emptive atom — for instance, with $v = (2, 0, 1)$. More subtly, the slope ϕ' does not by itself determine the current timing game, because the relation of marginal and average rewards, ϕ and Λ, is critical. Pre-emptive atoms subsume intervals when ϕ is decreasing, by Theorem 2-(a); hence the atom is larger than necessary to reach a level of G so that $\phi' > 0$; we thus say that the atom is 'inflated' relative to an atom that would be prescribed by the naïve direct-sum. The reverse, i.e. inflation of war of attrition phases, does not occur, as we now flesh out.

Theorem 3 (Truncation and Atom-Inflation) *Pre-emptive atoms are inflated and wars of attrition truncated: Any pre-emptive atom subsumes at least some portion of the adjacent intervals where ϕ is increasing, and where a war of attrition is played.*

Proof: A linear portion of a potential function Γ must be a common tangent to distinct convex portions of Φ, and corresponds to a pre-emptive explosion. If this tangent joins non-adjacent convex portions, then the atom is strictly inflated, as it subsumes at least one entire war of attrition phase. It therefore suffices to consider a common tangent τ of adjacent convex portions. Without inflation, such τ must touch at consecutive points of inflection of Φ, i.e. where $\phi'(g) = 0$. This is impossible, as it would slice through Φ. □

For instance, in the Caller Number Two of Three example, at most one phase transition occurs, since ϕ' changes its sign just once, from positive to negative when $g = 1/2$. Observe that the ODE defining the war of attrition is defined until time $t = 1/2$. While this may be its natural termination point, terminal atomic rewards are too small at that moment. Indeed, the atom would have size $G(1/2) = 1/2$, and $\Lambda(1/2, 1) = 1/3 < \phi(1/2) = 1/2$. Hence, the atom advances until $\phi(g)$ and $\Lambda(g, 1)$ cross. This occurs when $\Lambda(g, 1)$ has a maximum at $g = 1/4$, i.e. $G(3/8) = 1/4$. This is before time $t = 1/2$, hence truncation.

4.2 The Number of Equilibria

We now find that the number of equilibria is potentially quite large — about two raised to half the number of phase transitions. Specifically, let $\mathcal{E}_m$ denote the set of symmetric stationary Nash equilibria, where m is the number of alternating slope signs of ϕ. Given the expected flow rewards ϕ, we can tie down the maximal cardinality of $\mathcal{E}_m$.[7]

Theorem 4 (How many equilibria?)
Assume ϕ has exactly m alternating slope-signs. Then the maximum number of equilibria $|\mathcal{E}_m|$ is $2^{|\mathcal{J}_m|}$, where $\mathcal{J}_m$ is the set of up-slopes of ϕ followed by down-slopes.

Proof: An equilibrium implies a unique set of up-slopes played (the common tangent on pairs of strictly convex portions of Φ is unique). Indeed, an initial down-slope prior to $\mathcal{J}_m$ does not affect the number of equilibria, as the down-slope is skipped in a jump. A terminal up-slope likewise does not affect the number of equilibria. It will either be skipped by a pre-emptive atom or played in a war of attrition, but not both. So there is a 1-1 map from equilibria $\mathcal{E}_m$ to sets $\mathcal{J}_m$ — hence, the power set enumeration for the upper bound of $|\mathcal{E}_m|$. □

The number of slopes m (up-down-…or down-up-…) is either odd or even. Suppose ϕ slopes up at $g = 0$. We then have to find the number of up-slopes followed by down-slopes: if m is even, this number k satisfies $m = 2k$; if m is odd then $m = 2k + 1$. The theorem states that the maximal number of equilibria is $|\mathcal{E}_{2k}|, |\mathcal{E}_{2k+1}| \leq 2^k$. Likewise, if ϕ slopes down at $g = 0$, then when m is odd, the number of up-slopes followed by down-slopes satisfies $m = 2k - 1$ so that $|\mathcal{E}_{2k-1}|, |\mathcal{E}_{2k}| \leq 2^{k-1}$.

For instance, the standard war of attrition has one slope-sign, and thus has $|\mathcal{E}_{2\cdot 0+1}| = 2^0 = 1$ equilibrium. The U-shaped game has two slopes, but slopes down first, so that it has at most $|\mathcal{E}_{2\cdot 1}| = 2^{1-1} = 1$ equilibrium. Caller Number Two of Three has $m = 2$ slopes, and exactly one up-slope followed by a down-slope, so that there are maximally $|\mathcal{E}_{2\cdot 1}| = 2^1$ equilibria.

For *Zick-Zack*, the theorem asserts that the terminal up-slope should not affect the maximum number of equilibria, i.e. still $|\mathcal{E}_{2\cdot 1+1}| \leq 2^1$. Why? Clearly, if $\psi \leq 1$, then the unique equilibrium is a war of attrition. If $\psi > 1$, then Φ has two points of inflection, and there are three possible potential functions: The first begins with a linear segment τ_0 that touches the second convex portion of Φ and is then strictly convex. The second is strictly convex, ending with a linear portion through $(1, \Phi(1))$. This linear segment τ_1 is tangent to the first convex portion of Φ and must have slope $\Gamma'(1) \geq \Phi'(1)$. The

[7] For a recent contribution on the number of NE in Normal form games see McLennan (2005).

last potential function has a linear segment τ in the interior of $[0,1]$ which is the unique common tangent to the first and second convex portions of Φ.

By construction, each of these potential functions is unique — if it exists. Observe that the tangent τ necessarily first touches Φ at some $g \in (0,1)$, because $\Phi'(0) = \phi(0) = 0 < \phi(g) = \Phi'(g)$ for $g > 0$. However, its second touch point occurs at some interior $h < 1$ only in some conditions, namely iff $\psi \in [\underline{\psi}, \bar{\psi})$. Moreover, as is geometrically clear, the tangents τ and τ_1 coincide at the very moment that $\psi = \bar{\psi}$. The tangent τ_1 in fact exists for $\psi \geq \underline{\psi}_1 := (11 + 3\sqrt{17})/16$. But its slope only weakly exceeds $\Phi'(1)$ for $\psi \geq \bar{\psi}$, where $\bar{\psi} > \underline{\psi}_1$. Altogether, τ_1 is part of a potential function iff $\psi \geq \bar{\psi}$.

This illustrates why the terminal up-slope in Zick-Zack does not increase the number of equilibria relative to the Caller Number Two of Three game: tangent τ_1 represents a terminal atom skipping the last up-slope, while τ corresponds to an interior atom after which the terminal up-slope is played. Precisely one of the two obtains.

One can finally show that the initial tangent τ_0 exists for $9/5 := \underline{\psi}_0 \leq \psi \leq 3 := \bar{\psi}_0$. For $\psi > \bar{\psi}_0$, τ_0 is no longer tangent to the second convex portion of Φ. For when $\psi = \bar{\psi}_0$, τ_0 becomes a straight line from the origin to $(1, \Phi(1))$ corresponding to a time zero unit atom. In summary, the maximum number of equilibria (two) is attained iff $\psi \geq \underline{\psi}_0$.[8]

When is the maximum number of equilibria attained?[9] One may be tempted to think it sufficient that $\mathsf{vex}(\Phi)$ touches all convex portions of Φ, as in Theorem 2-(c). But the above analysis of Zick-Zack shows that this is not enough: For $\psi \in [\underline{\psi}, \underline{\psi}_0)$, $\mathsf{vex}(\Phi)$ touches both convex portions of Φ, and yet the induced equilibrium is unique.

Even when the maximal number of equilibria is attained, no equilibrium need attain the maximal number of phase transitions. In Zick-Zack, both equilibria have only one phase transition for $\psi > \bar{\psi}$, while the most phase transitions is two, by Theorem 2-(c).

If we drop the assumption that 0 belongs to the support, then a continuum of equilibria may arise: Suppose that in the set of equilibria that we identify there is one with an atom at time zero and $\Gamma'(0) > \phi(0)$, as occurs in Caller 2 of 3. Then there is a maximum time t such that $e^{-rt}[\Gamma'(0) - c(t)] = \phi(0)$. Then for every $s \in (0, t]$, play according to Γ starting at time s is an equilibrium.

[8]The $\underline{\psi}_0, \underline{\psi}_1$ thresholds are most easily obtained via $\Lambda(0,g)$ and $\Lambda(g,1)$: First, $\Lambda(0,g)$ has an interior maximum and minimum for all $\psi \geq \underline{\psi}_0$, and is monotonic for smaller ψ. If a potential function starts with a linear portion, then τ_0 is tangent to $\Phi(g)$ exactly when $\phi(g)$ and $\Lambda(0,g)$ intersect at an interior minimum of $\Lambda(0,g)$. The middle panel of Figure 4 illustrates this point. The threshold $\bar{\psi}_0$ for ψ allows $\phi(g)$ and $\Lambda(0,g)$ to cross at $g = 1$. The computations for $\underline{\psi}_1$ follow similar lines of reasoning using $\Lambda(g,1)$. Finally, the payoff from the interior maximum of $\Lambda(g,1)$ coincides with $\phi(1)$ at $\bar{\psi}$.

[9]Detailed sufficient conditions for this are available from the authors upon request.

5 Equilibrium Payoffs

We now ask what is each player's expected payoff, and how much "rent" is lost by delay? In the unobserved actions pure war of attrition, the (common) expected payoff is the initial flow reward $\phi(0) = v(1)$, and all rents are dissipated, namely the difference $\phi(1) - \phi(0) = v(N+1) - v(1)$ between highest and lowest rank payoffs — the total variation in rank payoffs. But with non-monotonic rank payoffs, the total variation of rank-payoffs is no longer the tightest bound on payoff dissipation. We are now forced to make a stronger assumption about delay costs. We assume no discounting and constant marginal participation costs, $c(t) = t$, so that rent dissipation coincides with the length of play.

Theorem 5 (Payoffs) *Assume no discounting and linear participation costs $c(t) = t$.*
(a) Fix an equilibrium corresponding to a given potential function Γ. Then the expected payoff is $\Gamma'(0)$, and the game must end after an elapse time of $\Gamma'(1) - \Gamma'(0)$.
(b) The equilibrium with the least expected payoff and maximal length corresponds to $\mathsf{vex}(\Phi)$. Thus, the least value is $\mathsf{vex}(\Phi)'(0)$, and the greatest length is $\mathsf{vex}(\Phi)'(1) - \mathsf{vex}(\Phi)'(0)$.

For a given potential function, the equilibrium expected payoff of the game is a local minimum of the forward looking average rewards; the game lasts until a local maximum of backward average rewards obtains. Moreover, the least expected payoff of the game is the global minimum of the forward average payoffs, and the maximal time elapse likewise occurs when the global maximum backward average rewards are reached.

Proof of (a): Fix a potential function Γ. Since the mixed strategy ensures a constant payoff along the support of play, the expected payoff of the game is the time zero payoff $\Gamma'(0)$. By Theorem 1 the game ends in finite time. The length of play depends on the payoffs dissipated — the higher the payoff they can obtain, the longer people are willing to delay. Since expected rank-payoffs must increase along the support of play, the largest rank-payoff $\Gamma'(1)$ obtains when the game ends.

Proof of (b): Suppose, counterfactually, that $\Gamma'(0) < \mathsf{vex}(\Phi)'(0)$ for some potential function Γ. Since $\mathsf{vex}(\Phi) \leq \Phi$ everywhere, we have $\mathsf{vex}(\Phi)'(0) \leq \Phi'(0)$, and thus $\Gamma'(0) < \Phi'(0)$. Then Γ is initially linear by $(P3)$. But differentiability and $(P3)$ jointly imply that Γ can only change slopes while tangent to Φ. If this happens at $g \in (0,1)$, then $\mathsf{vex}(\Phi)(g) \leq \Phi(g) = \Gamma(g) = \Gamma'(0)g < \mathsf{vex}(\Phi)'(0)g$. This violates convexity of $\mathsf{vex}(\Phi)$.

Similarly, at $g = 1$ we have $\Gamma'(1) \leq \mathsf{vex}(\Phi)'(1)$ for any potential function Γ. □

This result extends the standard war of attrition with monotonic rank rewards: When ϕ is monotonic, Φ is globally convex, and the only potential function is Φ itself. The expected payoff is $\Phi'(0) = \phi(0) = v(1)$, and the maximal length is $\Phi'(1) - \Phi'(0) =$

$\phi(1) - \phi(0) = v(N+1) - v(1)$. In fact, by $(P3)$ and Theorem 5-(b), this is the length of any unobserved actions game where $\mathsf{vex}(\Phi)$ begins and ends on a strictly convex portion.

Since rank payoffs are smoothed in ϕ with unobserved actions, the total variation in $\phi = \Phi'$ is a tighter upper bound on payoff dissipation (eg. Figure 4, left). But war of attrition-phases are truncated, and even this measure is not tight enough. The length and expected payoff depend on the slopes of the initial and terminal tangents τ_0 and τ_1.

In Caller Number Two of Three, $\mathsf{vex}(\Phi)$ is strictly convex for $g \leq 1/4$ and linear with slope $3/8$ for $g > 1/4$. The expected payoff is $\phi(0) = 0$ and the maximum length of the game is $3/8$. In the U-shaped example, $\mathsf{vex}(\Phi)$ is linear with slope $5/8$ for $g < 3/4$ and strictly convex for $g \geq 3/4$. The expected payoff in the game is the first flow payoff in the war of attrition, $\phi(3/4) = 5/8$, and the maximum elapse time equals $\phi(1) - \phi(3/4) = 3/8$.

In Zick-Zack, with rank rewards $(0, \psi, 0, 1)$, $\mathsf{vex}(\Phi)$ is the potential function that starts with a strictly convex portion. Thus, the minimum expected flow payoff is $\phi(0) = 0$. For $\psi \leq \underline{\psi}$, Φ is strictly convex, and the unobserved actions game is then equivalent to a war of attrition. If $\psi \in (\underline{\psi}, \bar{\psi})$, $\mathsf{vex}(\Phi)$ touches both convex portions of Φ and hence $\mathsf{vex}(\Phi)'(0) = \phi(0)$ and $\mathsf{vex}(\Phi)'(1) = \phi(1)$. Thus, the maximum duration is $\phi(1) - \phi(0) = 1$, which is below the total variation ψ in rank payoffs. Finally, for $\psi > \bar{\psi}$, $\mathsf{vex}(\Phi)$ ends with a linear portion, and the terminal payoff is governed by the slope of the tangent τ_1. The maximum duration exceeds $\phi(1) - \phi(0)$, but is still less than the total variation of ϕ.

What about the *most efficient* equilibrium? If $\Phi(1) \geq \Phi'(1)$, then $\Gamma^*(p) = p\Phi(1)$ is a potential function, and clearly corresponds to a time-0 complete atom. But if $\Phi(1) < \Phi'(1)$, then a time-0 jump is no longer an equilibrium. In some of these cases, we can identify the most efficient equilibrium, but we have found no clear theorem. For there are examples where the equilibrium with the greatest expected payoff is not the quickest.

Assuming that $\Phi''(1-) = \phi'(1-) > 0$, for instance, if we can construct a tangent τ^* from the origin to the last convex portion of Φ, tangent at some $\bar{p} \in (0, 1]$, then it is the most efficient equilibrium by both measures: shortest and greatest expected payoff. The shortest equilibrium in Zick-Zack is induced by the potential function Γ with a linear segment at the origin; such a potential function exists when $\psi \geq \underline{\psi}_0$. Since $\Gamma \neq \mathsf{vex}(\Phi)$, its expected payoff is higher. Also, for $\psi < \bar{\psi}_0$, its terminal slope is $\Gamma'(1) = \Phi'(1) = \phi(1) = 1$, which is weakly smaller than $\mathsf{vex}(\Phi)'(1)$. Thus, it is the shortest equilibrium. For $\psi \geq \bar{\psi}_0$, the atom is complete; this equilibrium is the shortest with the maximal expected payoff.

6 Conclusion

The timing game literature has long partitioned into wars of attrition and pre-emption games. The incentive structure for both varieties of timing games finds a common home in this paper. This paper introduces the idea of potential functions into this class of timing games, using them to characterize the stationary symmetric Nash equilibria. This has afforded a strikingly quick existence proof, and tractable analysis of these equilibria. The resulting equilibria are remarkably rich, with on-path atomic explosions that may be preceded or followed by slow wars of attrition. Further, the two types of timing games interact with each other, with anticipation of later phases influencing current play. Thus, the moments for the explosions are advanced in time relative to a naïve "direct sum".

Exogenous payoff growth over time, a feature often associated with pure pre-emption games, is an obvious extension that we pursue in other work.

Observed Actions. This paper is a first step to understanding the important class of timing games with non-monotonic payoffs. Its success has hinged on an assumption of unobservable actions. While full observability of actions is equally unreasonable, settings with observable actions are seen as more typical. Radio call-in shows aside, there are real-world situations with unobserved actions. For instance, historically, so-called "silent" duels (one class of timing games) were extensively studied, and for good reasons: the development of weapons systems (e.g. submarines, stealth bombers) is a situation where one party's actions remain hidden from counterparties, at least for long stretches of time. Abreu and Brunnermeier (2003) recently model financial market trading as an unobservable actions timing game. Market entry decisions, as argued before, are often taken long in advance; these decisions are also taken with unobserved actions.

Characterizing equilibrium behavior a timing game with many players is difficult, as can also be seen with LEV who, despite their insightful existence results, have spelled out no characterizing results. Faced with a tough problem, one tackles the simplest version of it first — here, that means one information set. In Appendix A, we briefly argue that our insights extend to observable actions.

A Appendix: Lessons for Observable Actions

Our theory assumes unobserved actions because the resulting analysis is tractable — but we do believe that it captures many economic situations. Below, we argue that it also provides a benchmark for understanding behavior with observable actions.

Once actions are observed, the model grows substantially more complex. Subgame-perfect equilibrium (SPE) is the mandated solution concept. Since players can see the game unfolding, there are now multiple information sets, one for each number of remaining players. There are therefore far more equilibria, since the number of remaining players itself can serve as a coordination device. We shall thus confine attention to symmetric SPE for which players engage in a war of attrition whenever possible, and a pre-emption game only when necessary. This substitutes for the stationarity condition for Nash equilibrium. For intuitively, a pre-emptive atom requires a high degree of coordination, and a war of attrition needs no coordination at all. Despite this refinement which seeks to minimize the role of pre-emption games, we now argue that our main qualitative finding still obtains: wars of attrition are truncated, and pre-emption atoms inflated.

Let $w(k+1)$ be the expected SPE payoff from the subgame after k have stopped.

Lemma 5 *A war of attrition obtains if $v(k+1) < w(k+2)$ while a pre-emptive atom of some size $p \in (0,1]$ occurs if $v(k+1) \geq w(k+2)$.*

Proof: Any $p < 1$ must equate the expected flow payoffs from the continuation game and the (shared) atomic payoffs:

$$\sum_{i=0}^{N-k} \binom{N-k}{i} p^i (1-p)^{N-k-i} w(k+1+i) = \sum_{i=0}^{N-k} \binom{N-k}{i} p^i (1-p)^{N-k-i} \mathsf{A}(k,i). \qquad (5)$$

Now, the LHS of (5) is flatter than the RHS of (5) at $p = 0$. For comparing slopes yields:

$$(N-k)(v(k+2)-v(k+1))/2+(N-k)(w(k+2)-v(k+1)) < (N-k)(v(k+2)-v(k+1))/2$$

since $w(k+2)-v(k+1) < 0$. Both sides are continuous in p and coincide for $p = 0$. Thus, they either intersect again for some $p \leq 1$, or, if not, the RHS atomic payoff dominates the LHS continuation payoff for all p, and a complete atom must obtain. □

Assuming again a constant cost of delay $c(t) = t$, the expected length of the war is $w(k+2) - v(k+1)$, while its expected payoff is $v(k+1) =: w(k+1)$. Assume that rank payoffs rise from j to k. We say that a war of attrition is *truncated in time* if its expected duration is less than $v(k) - v(j)$. Call a war of attrition *weakly truncated* (i.e. in ranks) if it nowhere obtains in $\{j, \ldots, k\}$, or if it obtains from j' to k' for some $j \leq j' < k' \leq k$. Likewise, if rank payoffs fall from j to k, the pre-emption game is *weakly inflated* (in ranks) if it obtains from j' to k' for some $j' \leq j$ and $k' \geq k$. Once an atom occurs, there is further atomic entry until a war of attrition-subgame is reached.

(♢) All rank payoffs on down-slopes are more valuable than the overall average remaining payoff, or $v(k+1) > \mathsf{A}(k, N-k)$ whenever $v(k+1) < v(k)$, for any k.

Theorem 6 *Assume* ($\diamondsuit$). *Wars of attrition are truncated in time, weakly truncated in ranks and pre-emptive atoms are weakly inflated.*

PROOF: As players are symmetric, they cannot expect to gain more than the average remaining rank payoff, $w(k+1) \le \mathsf{A}(k, N-k)$. So ($\diamondsuit$) implies $v(k+1) > w(k+1)$.

A war of attrition along an up-slope from a minimum rank $\underline{k}$ to $\bar{k}$ lasts at most time $w(\bar{k}) - v(\underline{k})$; it is thus truncated in the *time* dimension from the naïve length $v(\bar{k}) - v(\underline{k})$.

Atomic entry obtains whenever $v(k) > w(k+1)$. Assume that there are subsequent up-slopes of rank-rewards. If the atom is complete, then it is clearly inflated. If the atom is incomplete, then with positive probability play continues on the same down-slope. But then $v(k) > v(k+1) > w(k+1)$, and another atom follows immediately. So once atomic entry starts, it stops only when play begins weakly on an up-slope. □

Corollary *Assume* ($\diamondsuit$). *The lowest expected equilibrium payoff with unobservable actions,* $\mathsf{vex}'(\Phi)(0)$, *is a lower bound for the expected payoff in an observable actions setting.*

This corollary can, of course, also be applied to every subgame of the observable actions setting, where $\mathsf{vex}'(\Phi)(0)$ is to be computed for the unobservable actions game with $N + 1 - k$ players. The corollary is a direct consequence of $\mathsf{vex}'(\Phi)(0)$ constituting a global minimum of the right hand side of equation (5). It is not true, however, that $\mathsf{vex}(\Phi)'(1) - \mathsf{vex}(\Phi)'(0)$ is some kind of bound on the elapse time: this is due to the fact that rank payoffs become left truncated as people stop. Hence $\mathsf{vex}(\Phi)'(1)$ has no direct counterpart relation in a setting with observable actions.

Corollary *Assume* ($\diamondsuit$). *There are at most as many phases as slope signs of* $v(k)$.

B Appendix: Omitted Proofs

B.1 Equilibrium Structure: Proof of Lemma 1

Let t be in the interior of the support of G. We prove that if G does not jump at t, then G is differentiable at t. Since payoffs (2) are constant on the support, $e^{-rt}(\phi(G(t)) - c(t)) =: \psi > 0$ is a constant when $G(t) < 1$. Since ϕ is a degree N polynomial, $\phi' = 0$ at most $N-1$ times, between which ϕ' is positive or negative. There are three cases to consider.

CASE 1. If $\phi'(G(t)) > 0$ at t, then G is differentiable at t, with

$$\dot{G}(t) = \frac{\dot{c} + r\ (\phi(G(t)) - c)}{\phi'(G(t))}.$$

CASE 2. If $\phi'(G(t)) < 0$ at t, then $\phi(G(t) - \epsilon) > \phi(G(t)) > \phi(G(t) + \epsilon)$ for all small enough $\epsilon > 0$. Since t is inside the support of G, but is not in an atom, there exists $\delta > 0$ with $\phi(G(t) - \epsilon) > \phi(G(t - \delta)) > \phi(G(t)) > \phi(G(t + \delta)) > \phi(G(t) + \epsilon)$. Since $e^{-r(t-\delta)} > e^{-rt}$ and $c(t - \delta) < c(t)$, a constant payoff is impossible because

$$e^{-r(t-\delta)}(\phi(G(t - \delta)) - c(t - \delta)) > e^{-rt}(\phi(G(t)) - c(t)).$$

In other words, $\phi'(G(t)) < 0$ cannot obtain in equilibrium.

CASE 3. Suppose $\phi'(G(t)) = 0$ at t. If this is a saddle point with $\phi' < 0$ left and right of $G(t)$, then $G(t) = \phi^{-1}(c(t) + e^{rt}\psi)$ locally. But this is decreasing, and so not a solution. Otherwise, $\phi' > 0$ is increasing on at least one side of t, where $G(t) = \phi^{-1}(c(t) + e^{rt}\psi)$ locally describes the unique smooth solution of the ODE.

Finally, if $t = 0$ or if the support interval of G is $[0, t]$, then the argument that G is differentiable (right or left, respectively) is a slight modification of the above analysis.

B.2 Potential Function Equivalence: Proof of Lemma 3

Fix a potential function Γ. By $(P3)$, $[0, 1]$ partitions into subintervals with endpoints $0 = \xi_0 < \xi_1 < \cdots < \xi_k = 1$, where[10] $\Gamma = \Phi$ or Γ is linear on alternating $[\xi_i, \xi_{i+1}]$.

First, $\Gamma(\xi_i) = \Phi(\xi_i)$ for all i, by $(P1)$ or $(P3)$. Assume Γ is linear on $[\xi_i, \xi_{i+1}]$. Then

$$\Lambda(\xi_i, \xi_{i+1}) \equiv \frac{\Phi(\xi_{i+1}) - \Phi(\xi_i)}{\xi_{i+1} - \xi_i} = \frac{\Gamma(\xi_{i+1}) - \Gamma(\xi_i)}{\xi_{i+1} - \xi_i} = \begin{cases} \Gamma'(\xi_{i+1}) & = & \Phi'(\xi_{i+1}) & \text{if } \ \xi_{i+1} < 1 \\ \Gamma'(\xi_i) & = & \Phi'(\xi_i) & \text{if } \ \xi_i > 0 \end{cases}$$

by smoothness $(P2)$ and Lemma 2. So $(E2)$ obtains: Stoppers earn identical payoffs just *before* atomic entry if $\xi_i > 0$, for then $\Lambda(\xi_i, \xi_{i+1})$ equals $\Phi'(\xi_i) = \phi(\xi_i)$, and *after* atomic entry if $\xi_{i+1} < 1$, since $\Phi'(\xi_{i+1}) = \phi(\xi_{i+1})$. Also, $(E1)$ holds, as flow payoffs are positive, by $\Gamma(\xi_{i+1}) > \Gamma(\xi_i)$. If $\xi_{i+1} = 1$, then $\phi(1) = \Phi'(1) \leq \Gamma'(1) = \Lambda(\xi_i, 1)$ by $(P1)$; so $(E3)$ holds.

Assume $\Gamma = \Phi$ on $[\xi_i, \xi_{i+1}]$, so that $\phi = \Phi' = \Gamma'$ (which exists by $(P2)$). We then needn't worry about $(E3)$. Since Γ is convex by $(P2)$ and Φ smooth, $\phi' = \Gamma'' \geq 0$. Also, ϕ is strictly increasing inside the interval, being a nonconstant polynomial; thus, $(E1)$ holds, as ϕ can only initially vanish. Assume that $G(\underline{t}) = \xi_i$, for some $\underline{t} \geq 0$. Thus, the ODE $\dot{G} = (\dot{c} + r[\phi(G) - c])/\phi'(G)$ in (3) admits the "constant payoff" solution $e^{-rt}[\phi(G(t)) - c(t)] = \phi(0) = \Gamma'(0)$, the initial payoff. (Recall that the support of G includes 0.) Hence, $(E2)$ obtains. Let $C(t) := c(t) + e^{rt}\Gamma'(0)$. Since ϕ is strictly increasing on (ξ_i, ξ_{i+1}), $G(t) = \phi^{-1}(C(t))$ obtains on the domain $(\underline{t}, \bar{t})$, where $\bar{t} = C^{-1}(\xi_{i+1})$.

[10]By $(P3)$ alone, the number k of such intervals may be infinite; Theorem 2 will rule out $k = \infty$.

Next, fix an equilibrium $G(t)$. For $(P3)$ to hold, the potential function inducing this equilibrium is found via: $\Gamma(g) = \Phi(g)$ whenever $G(t)$ is continuous at $G^{-1}(g)$; at any jump from g to h, Γ is the linear function through $(g, \Phi(g))$ and $(h, \Phi(h))$, with slope

$$\frac{\Gamma(h) - \Gamma(g)}{h - g} = \frac{\Phi(h) - \Phi(g)}{h - g} \equiv \Lambda(g, h) \begin{cases} \geq \phi(h) & \text{with equality if } h < 1 \\ = \phi(g) = \Gamma'(g) & \text{if } g > 0 \end{cases} \tag{6}$$

by constant payoffs $(E2), (E3)$. This gives $(P2)$: Γ is differentiable, increasing (by (6) or by $\Gamma' = \phi > 0$), and convex: either Γ is linear, or has slope ϕ, increasing by $(E2)$.

Finally, we show $(P1)$. If $\Gamma = \Phi$ near 1, then $\Gamma(1) = \Phi(1)$ and $\Gamma'(1) = \Phi'(1)$. If $\Gamma = \Phi$ near 0, then $\Gamma(0) = \Phi(0)$. If $G(t)$ starts with a jump from 0 to h, then Γ has a linear segment with slope $\Phi(h)/h$ through $(h, \Phi(h))$. This forces $\Gamma(0) = 0$. If G ends with a jump to 1, then $\Gamma'(1)$ is the final linear slope, i.e. $\Gamma'(1) \geq \phi(1) = \Phi'(1)$ by (6). □

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